

# Scalar Material Reference Systems and Loop Quantum Gravity

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## Abstract

In the past, the possibility to employ (scalar) material reference systems in order to describe classical and quantum gravity directly in terms of gauge invariant (Dirac) observables has been emphasised frequently. This idea has been picked up more recently in Loop Quantum Gravity (LQG) with the aim to perform a reduced phase space quantisation of the theory thus possibly avoiding problems with the (Dirac) operator constraint quantisation method for constrained system.

In this work, we review the models that have been studied on the classical and/or the quantum level and parametrise the space of theories so far considered. We then describe the quantum theory of a model that, to the best of our knowledge, so far has only been considered classically. This model could arguably called the optimal one in this class of models considered as it displays the simplest possible true Hamiltonian while at the same time reducing all constraints of General Relativity.

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# 1 Introduction

In series of seminal papers [1, 2, 3, 4], Kuchař and his collaborators have constructed a whole class of manifestly spacetime diffeomorphism invariant matter actions which have the remarkable feature to clarify the conceptual setup of General Relativity. A further model of this type, originally introduced by Rovelli and Smolin in [5], was analysed by Kuchař and collaborators in [6]. The motivation for introducing those matter actions is to use the matter considered as a material reference system. This allows to isolate the pure gauge degrees of freedom (under spacetime diffeomorphisms) encoded in general relativity in an elegant way and yields to theories where gauge invariant (Dirac) observables can be constructed by means of those matter reference fields. The matter considered in [1, 2, 3] was coined dust matter because its energy momentum tensor is that of a pressure free perfect fluid. The advantage of the availability of such a description of the phase space of matter and geometry with regard to quantum gravity was already emphasised in those works although a concrete quantum framework was not available at that time and the quantum theories following from this models could only be discussed at a formal level.

A particular feature of the matter models considered is that they lead to a deparametrised form of general relativity (plus, if considered, standard model matter) and that the resulting hypersurface deformation algebra for the Hamiltonian constraints, that is the algebra among the individual constraints, becomes Abelian. This can be achieved by writing those Hamiltonian constraints in an equivalent<sup>1</sup> form in which they are linear in the reference matter momenta. This is an important property because in this form the constraints form a true Lie algebra in contrast to the Hamiltonian constraints in its standard form. As far as the quantum theory is considered, this is an advantage because many techniques that aim at solving the constraints in the quantum theory (such as group averaging [7]) cannot be applied when there occur structure functions instead of structure constants.

The equivalent form of the Hamiltonian constraint that has mutually commuting Poisson brackets with itself is of the typical form  $C^{\text{tot}}(x) = P(x) + h(x)$  where  $P$  is the momentum conjugate to one of the dust fields  $T$  (called the clock field) and  $h$  is a scalar density of weight one built from the non dust, spatial scalar density weight one contribution  $C$  to the Hamiltonian constraint, the (square root of) the determinant  $Q = \sqrt{\det(q)}$  of the spatial metric  $q_{ab}$ , the density two spatial scalar  $D = q^{ab}C_a C_b$  where  $C_a$  is the non dust contribution to the spatial diffeomorphism constraint, the density one respectively zero spatial scalars  $V = q^{ab}T_{,a}C_b$ ,  $U = q^{ab}T_{,a}T_{,b}$ . For a certain subclass of models, the dependence of  $h$  on  $V, U$  is trivial and it then follows from the first class property of the constraint algebra that the  $h(x)$  themselves must be mutually vanishing.

In [8] the general solution of phase space functions  $h(x)$  of density weight  $w$  built from  $C(x), Q(x), D(x)$  (but not on  $V(x), U(x)$ ) with mutually commuting Poisson brackets was found in terms of a first order PDE which can be solved by the method of characteristics, thus extending the set of solutions that were obtained from the concrete models [1, 2, 3, 5, 6] but without providing concrete covariant actions from which they result. In [9] it was shown that all of these solutions can be obtained by using covariant actions of the Lagrange multiplier type used in [1, 2, 3, 4, 5, 6].

On the other hand, one can ask whether there are covariant scalar matter models that do not rely on Lagrange multipliers at all. This question was studied in [10] and it was pointed out that provided that the scalar field has pure derivative coupling to the gravitational field then such models again give rise to mutually Poisson commuting Hamiltonian densities  $h$ . Also the inverse question can be analysed using the methods developed in [10], namely, given a general solution  $h$  as described in [8] (with density weight one), what is the Lagrangian  $L$  that gives rise

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<sup>1</sup>I.e. they define the same constraint surface and the same Dirac observables.

to it? The solution, when it exists, can be written implicitly as the solution of an ODE.

The general framework for describing the reduced phase space of a constrained system with a generally covariant Lagrangian in manifestly gauge invariant form has been considered, to the best of our knowledge, for the first time in [11, 12]. Aspects of this have been rediscovered in different language by several authors, e.g. [13, 14, 15]. In these works, a complicated formula appears that “projects” a non gauge invariant function  $f$  on phase space to a (weakly) gauge invariant one. It involves an infinite series of multiple Poisson brackets between  $f$  and the Hamiltonian constraint which makes it apparently impractical to use. However, as pointed out in [16], the Poisson algebra of these formal observables can nevertheless still be simple, thus enabling in principle a concrete reduced phase space quantisation approach to quantum gravity.

In [17, 18] the frameworks of [1, 11, 12] were combined in order to compute the classical reduced phase space of [1]. Namely, in [1, 2, 3, 5, 6] the reduced phase spaces were only described for some models in suitable gauges and the last missing step before arriving at the fully reduced phase space was to use the projector formula provided in [11, 12] and construct general observables with respect to the Hamiltonian constraint.

The reduced phase space described in [17, 18] was then quantised using the technology of Loop Quantum Gravity [19, 20] in [21] and thus providing for the first time a concrete model for LQG where *all constraints are solved*. This means that all operators in question are *Dirac observables* and time evolution is driven by a *physical (i.e. non vanishing, gauge invariant) Hamiltonian* for which a concrete quantisation was provided. In [21] two possible quantisations of the reduced physical phase space were discussed. In the first one the usual kinematical Hilbert space of LQG becomes the *physical Hilbert space*, no solutions of the constraints have to be computed, the usual LQG inner product is the *physical inner product* here. In the second possibility techniques from the Algebraic Quantum Gravity framework introduced in [26] are considered and the quantum theory is defined on an abstract algebraic graph using von Neumann’s infinite tensor product Hilbert space.

Later, in [22], a scalar matter model of the type considered in [10], namely the model of [5, 6] was used in order to perform the reduced phase space quantisation by methods from LQG of General Relativity with a matter content different from [21]. While in [21] four scalar reference fields are considered the model in [22] contains only one scalar field. The most significant difference between these works is that the spatial diffeomorphism constraint is solved already classically in [21] using three out of the four reference fields while in [22] it needs to be solved at the quantum level.

A reduction of the diffeomorphism constraint at the classical level has the advantage that one can avoid ambiguities that can occur in the quantum reduction. These exist also for the spatial diffeomorphism constraint and are in general not unproblematic because those could possibly present an anomaly of the algebra of observables [23]. It turns out that the classical reduction with respect to the spatial diffeomorphism constraints of the dust model of [1] has the consequence, when using the standard LQG representation, that the physical Hamiltonian of the theory (a diffeomorphism invariant quantity) needs to be implemented as an operator that preserves the underlying graph on which the quantum states are defined<sup>2</sup>. If one wants to avoid a graph-preserving quantisation which was one of the motivations for the quantum model introduced in [22], then within LQG one is forced to solve the spatial diffeomorphism constraint at the quantum level as shown and implemented for the first time in [25]. In this case the operators involved are not knot class preserving.

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<sup>2</sup>This simply follows from the result that spatially diffeomorphism invariant operators that are graph-modifying are not densely defined in the standard kinematical representation of LQG as shown in [31]

As far as semiclassical aspects of the models are concerned, there are two issues to be kept in mind. First of all, semiclassical tools have not been developed at the level of the Hilbert space of spatially diffeomorphism invariant states at all, neither in the knot class preserving case nor in the non preserving case. Secondly, at the kinematical level, only for graph-preserving quantisations do exist semiclassical tools. These have been used to establish the correctness of the semiclassical limit of the dynamics for short time scales in [21] using techniques developed in [26]. However, for graph-modifying dynamical operators that occur the model in [22] one first needs to develop semiclassical states first for diffeomorphism invariant states to begin with and then in addition for knot class modifying operators before the semiclassical sector of the dynamics can be investigated. Notice that on the other hand in the Algebraic Quantum Gravity (AQG) framework of [26] here is no need to preserve algebraic subgraphs on which the algebraic spin network functions depend.

Another more recent model discussed in [24] that was also already described in the appendix of [1] as a special case of the model in [1] results by switching off three Lagrange multiplier fields in the general Lagrangian of [1] which leads to a model with only one Lagrange reference matter field and likewise to [22] deparametrises only the Hamiltonian constraint but does not rewrite the spatial diffeomorphism constraints in equivalent form that they are linear in the momenta of the reference fields as in [21].

The authors nevertheless claim that they can treat their model with the methods from [26] without having to care about the spatial diffeomorphism constraint anymore and nevertheless describe the physical sector of the theory. Moreover, they claim that they have found the most promising classical foundation for the reduced phase space quantisation of General Relativity using methods from LQG in the sense that the physical Hamiltonian simplifies in the maximal sense. As we will see, some of these claims are too strong.

It transpires that there are several models of General Relativity, the standard model matter and certain additional matter in the literature which lead to different reduced phase spaces and different physical Hamiltonians with different properties. It would therefore be desirable to start from a common platform in the form an action principle, to study the different models from a unified viewpoint and to analyse their different physical properties and promises. This is the objective of the present paper.

This paper is architected as follows:

In section two we review the known results from the literature and then construct a general Lagrangian that encompasses and extends all models considered so far in the literature with minimal coupling of the metric (no derivatives) to matter. It serves as the desired common platform or parametrisation of theory space. Different models can be reached by switching on and off certain parameters. Basically the models considered so far fall into two types: I. Those which just deparametrise time and II. those which deparametrise all of spacetime. We describe their Hamiltonian analysis in a unified fashion.

In section three we reconsider the model of type II studied in [24] and iterate arguments well known in the literature (e.g. [27, 28] and references therein) that reveal that some of the claims made in [24] are not tenable. It is not possible to simply drop the spatial diffeomorphism constraint and quantising using the algebraic quantum gravity framework as claimed by the authors. Rather one has to treat the quantisation of the dynamics in that model more carefully. A possible way to deal with the spatial diffeomorphism constraint in the algebraic framework is to quantise it by the Master constraint method, see [27] for a discussion, if one wants to pro-

ceed along [26]. In [22] the reduction of the spatial diffeomorphism constraints at the quantum level was performed via the refined algebraic quantisation technique [7] by means of a rigging map yielding the usual (spatially) diffeomorphism invariant Hilbert space of LQG. The physical Hamiltonian was then quantised on the diffeomorphism invariant Hilbert space.

Moreover, a careful treatment shows that there is an important issue with the choice of sign of a square root that has not been mentioned in [24]. Namely, if one insists on the usual energy conditions for the energy momentum tensor, then the sign of the physical Hamiltonian must be constrained in that model whence the physical Hamiltonian is still not without a square root as claimed, since it is constrained to equal plus or minus its absolute value. We also show that the model in [24] corresponds to a subsector of the model [21] which also explains why there is still an absolute value (and thus a square root) involved in the physical Hamiltonian of [24]. The fact that this is a subsector of [21] corresponding to vanishing spatial diffeomorphism constraint makes it also transparent that the latter cannot be dropped.

In section four we work out the details of the LQG quantisation of the model [3] following the procedure of [17, 18, 21]. This model could arguably called the optimal one in the class of models considered so far in the sense that not only also the spatial diffeomorphism constraint is already reduced classically, thus avoiding the ambiguities and potential problems pointed out in [23] in connection with the scalar product on spatially diffeomorphism invariant states, but furthermore, this time there is really no square root involved in the physical Hamiltonian.

In section five we summarise and conclude.

## 2 Theory Space

There are roughly two types of scalar field models that have been studied in the literature and that were found convenient for investigations in quantum gravity. Models of type I are based on Lagrangians that involve beside the fields that play the role of the reference fields additional fields that couple to gravity without any derivatives. We will call the latter Lagrange multiplier fields. Models of type II do not use Lagrange multiplier fields but consider only reference matter fields in their Lagrangians. Note that both types consider only scalar fields that are minimally coupled to gravity and hence the model of a conformally coupled scalar field in [29] is not considered in the classification here.

Common to both types of Lagrangians is that the non Lagrange multiplier fields enter the Lagrangian without a potential, that is, it depends only on the first derivatives of the fields. Those fields are used to construct the gauge invariant expressions of the remaining fields (geometry and standard matter) and as far as the final observables are constructed those fields have been absorbed. This can be understood likewise to the case of the Higgs mechanism of the standard model where three out of the four Higgs fields are 'eaten' by the boson fields yielding to a gauge invariant description of massive vector bosons. The main difference to the standard Higgs mechanism is that here all reference field considered in the models are 'eaten' by the other degrees of freedom, whereas in the ordinary Higgs mechanism one scalar field remains in the theory and is not absorbed. The reason why no potentials of the reference fields are considered is that one wants the physical Hamiltonian to be independent of the additional scalar fields introduced because those will play the role of the reference time and spatial reference points. Hence without a potential, one achieves that the physical Hamiltonian is then not explicitly dependent of the reference time and thus the reduced physical system becomes conservative which has obvious advantages. In principle, however, also potential terms could be treated if one gives up on the requirement of a conservative system. We will review both types of models I and II below and point out some of their most important physical differences.

## 2.1 Type I

The first class of models are those considered in [1, 2, 3] and we will briefly discuss their Lagrangians here. The famous Brown – Kuchař timelike dust Lagrangian [1] is given, in our notation, by

$$\mathcal{L}_{\text{TD}} = -\frac{1}{2}\sqrt{|\det(g)|} \rho [g^{\mu\nu} U_\mu U_\nu + 1], \quad U_\mu = \nabla_\mu T + W_j \nabla_\mu S^j \quad (2.1)$$

which depends on two pairs of respectively 4 scalar fields ( $\rho \geq 0, W_j$ ) and  $(T, S^j)$  respectively. Here latin indices  $j, k, \dots$  run from one 1 to 3 and greek indices  $\mu, \nu, \dots$  run from 0 to 3. We choose our signature convention for the spacetime metric tensor  $g_{\mu\nu}$  to be  $(-, +, +, +)$ . The Bičák – Kuchař null dust Lagrangian [2] is given by

$$\mathcal{L}_{\text{ND}} = -\frac{1}{2}\sqrt{|\det(g)|} g^{\mu\nu} U_\mu U_\nu, \quad U_\mu = W_j \nabla_\mu S^j \quad (2.2)$$

and results from (2.1) by setting  $\rho = 1$ ,  $T = 0$  and dropping the cosmological constant term  $-\rho\sqrt{|\det(g)|}/2$ . The opposite limit was taken in a discussion in the appendix of [1] where one sets  $W_j = 0$  and thus drops the dependence of (2.1) on  $S^j$  which results in

$$\mathcal{L}_{\text{NRD}} = -\frac{1}{2}\rho\sqrt{|\det(g)|} [g^{\mu\nu} U_\mu U_\nu + 1], \quad U_\mu = \nabla_\mu T \quad (2.3)$$

This possibility of non rotational dust can be understood as a special case of the timelike dust model.

The variables  $(\rho, T)$  and  $(W_j, S^j)$  appear symmetrically in (2.1) in the sense that if we write  $\lambda_0 := \sqrt{\rho}$ ,  $\lambda_j := \sqrt{\rho}W_j$  and  $S^0 := T$  then (2.1) becomes

$$\mathcal{L}_{\text{TD}} = -\frac{1}{2}\sqrt{|\det(g)|} [g^{\mu\nu} U_\mu U_\nu + 1], \quad U_\mu = \lambda_\alpha \nabla_\mu S^\alpha \quad (2.4)$$

The Gaussian dust model suggested in [3] treats them unsymmetrically and can be written in the form

$$\mathcal{L}_{\text{GD}} = -\frac{1}{2}\rho\sqrt{|\det(g)|} [g^{\mu\nu} (\nabla_\mu T) (\nabla_\nu T) + 1] - \sqrt{|\det(g)|} g^{\mu\nu} (\nabla_\mu T) V_\nu, \quad V_\mu = W_j \nabla_\mu S^j \quad (2.5)$$

## 2.2 Type II

The first type of model of this class is simply the massless Klein-Gordon action considered in [5] and whose canonical structure was discussed in detail in [6]. It leads, for general reasons that we review below, to an Abelian algebra of Hamiltonian constraints. As pointed out in [10], the mechanism responsible for this is by far not restricted to the Klein Gordon action but works for a general Lagrangian of the form

$$\mathcal{L}_S = \sqrt{|\det(g)|} L(I), \quad I = -\frac{1}{2}g^{\mu\nu} [\nabla_\mu T] [\nabla_\nu T] \quad (2.6)$$

where  $L$  is any function of the argument indicated. The first observation is that (2.6) in contrast to the type I Lagrangians depends on only a single scalar field  $T$  rather than several ones  $T, S^j$ . The reason for this is that when one simply adds Lagrangians of the type (2.6) or considers more generally Lagrangians depending on  $L(I_1, \dots, I_N)$  where  $I_k$  is as in (2.6) but  $T$  replaced by a field  $T_k$ , then the physical Hamiltonian no longer deparametrises. It is for this reason that the models of type I either do not depend on the  $S^j$  or if they do then only in the combination  $V_\mu = W_j \nabla_\mu S^j$  involving the Lagrange multipliers. We will see below why this is the case.

In any case, when we assume that observables are constructed using the reference matter fields and no geometric degrees of freedom we see that models of type II in contrast to some models of type I cannot lead to a fully reduced phase space at the classical level but only a reduction with respect to the Hamiltonian constraint can be performed.

### 2.3 Poisson commuting Hamiltonian constraints

One particular feature of the models in [1, 2, 3, 4, 5, 6] is that they all involve Hamiltonian constraints that satisfy an Abelian algebra. In the models in [1, 5, 6] one is furthermore able to rewrite the Hamiltonian constraints in deparametrised form, that is in the form  $P+h=0$  where  $P$  is the momentum conjugate to the time reference field and  $h$  is a function of all variables but the reference fields of density weight one, also called Kuchař density. As a consequence it follows immediately that also the Kuchař densities  $h$  mutually commute, that is  $\{h(x), h(y)\} = 0$  for all points  $x, y$  in the spatial hypersurface. For the reason that in the model in [5, 6] the spatial diffeomorphism constraint is not rewritten in a form linear in momenta of reference field, the conclusion of strongly commuting Kuchař densities  $h$  does not immediately follow but the algebra could close weakly, that is up to certain combinations of the spatial diffeomorphism constraint. However, as proven in [6] the corresponding Kuchař densities  $h$  of this model also commute strongly.

Those Kuchař densities are particular functions built from  $Q = \sqrt{\det(q)}$ ,  $D = q^{ab}C_aC_b$ ,  $C$  where  $q_{ab}$  denotes the spatial three metric and  $C_a, C$  respectively the non dust contributions to the spatial diffeomorphism and Hamiltonian constraints respectively (provided that matter couples only to the metric but not to its derivatives so that the hypersurface deformation algebra holds [30]). In general, rewriting the Hamiltonian constraints in this deparametrised form involves solving algebraic equations which leads to branches of the phase space labelled by the choice of certain signs, in this sense the description is only local on phase space and restricted to a single branch.

The existence of the Kuchař densities begs for the question whether one cannot obtain all functions  $h$  without going through an action principle. This problem was solved in [8]. Following [8] we introduce the density zero scalars  $d = D/Q^2$ ,  $c = C/Q$  and display a density  $w$  Kuchař density in the form  $h = Q^w K(c, d)$ . Then the infinitely many equations  $\{h(x), h(y)\} = 0 \forall x, y$  reduces, using the hypersurface deformation algebra, to the single first order PDE [8]

$$\frac{w}{2}K \frac{\partial K}{\partial d} = d \left( \frac{\partial K}{\partial d} \right)^2 - \frac{1}{4} \left( \frac{\partial K}{\partial c} \right)^2 \quad (2.7)$$

A general integral can be obtained in terms of  $S = \ln(K)$  by writing (2.7) in the form

$$-\frac{w}{2} \frac{\partial S}{\partial d} + d \left( \frac{\partial S}{\partial d} \right)^2 = \frac{1}{4} \left( \frac{\partial S}{\partial c} \right)^2 =: a^2 \quad (2.8)$$

which allows to separate the variables  $S(c, d) = S_1(c) + S_2(d)$  and to reduce (2.8) to two simple quadratures which can be written in closed form and involve two arbitrary parameters. The complete integral can then be obtained by the envelope construction.

All solutions that were obtained by considering models of type I or II are of course described by (2.7). One can ask the converse question how to build a Lagrangian of type I or II respectively which reproduces a solution of (2.7). This was analysed in [9] and [10] respectively. In [9] it is shown that for all solutions of (2.7) a type I action involving a single scalar field  $T$  with non derivative coupling of the metric and a single Lagrange multiplier field  $\rho$  can be found reproducing it. Remarkably, all solutions of (2.7) satisfying certain reality conditions (such as the positivity of the kinetic term) can be obtained from an action of the form

$$\mathcal{L}'_{\text{NRD}} = -\frac{1}{2} \sqrt{|\det(g)|} [\rho g^{\mu\nu} U_\mu U_\nu + \Lambda(\rho)], \quad U_\mu = \nabla_\mu T \quad (2.9)$$

That is, it is of the non rotating dust type just that the cosmological constant term is allowed to be a general function of the Lagrange multiplier. It has to satisfy an ODE that is matched to the two ODE's in (2.8). By contrast, the class of solutions of (2.7) that one obtains from the

type II models is more restricted. One finds from the Legendre transform ( $P = \partial L / \partial \dot{T}$ ) the following expression for  $p^2 := (P/Q)^2$  as shown in [9]

$$p^2 := \left( \frac{P}{Q} \right)^2 = (L'(I))^2 (I + q^{ab} T_{,a} T_{,b}) = (L'(I))^2 \left( I + \frac{d}{p^2} \right) \quad (2.10)$$

In the second step we used the spatial diffeomorphism constraint and applied the Brown-Kuchař mechanism, that is using the relation  $q^{ab} T_{,a} T_{,b} = D/P^2$  in order to replace the second term in the brackets above. The Hamiltonian constraint is given by

$$C^{\text{tot}} = Q \left( c + \frac{p^2}{L'(I)} - L(I) \right) \quad (2.11)$$

where  $I$  solves (2.10). To check whether a given solution  $p = -K(c, d)$  of (2.7) corresponds to a Lagrangian  $L(I)$  we insert  $p = -K(c, d)$  into (2.10) and solve the resulting equation for  $c = c(I, d)$ . Then (2.11) must become the identity

$$0 = c(I, d) + \frac{K(c(I, d), d)^2}{L'(I)} - L(I) \quad (2.12)$$

which is an ODE for  $L$  provided the explicit dependence of (2.12) on  $d$  drops out.

## 2.4 Global Parametrisation of Theory Space

We now combine these known results and write down an action including up to 8 scalar fields and several parameters that describes all the models so far considered. Since the models of type II are contained in the set of models of type I in the fashion described we focus on the former set. Consider

$$\mathcal{L}_D = -\frac{1}{2} \sqrt{|\det(g)|} (g^{\mu\nu} [\rho (\nabla_\mu T) (\nabla_\nu T) + A(\rho) V_\mu V_\nu + 2B(\rho) (\nabla_\mu T) V_\nu] + \Lambda(\rho)), \quad V_\mu = W_j \nabla_\mu S^j \quad (2.13)$$

where  $A, B, \Lambda$  are arbitrary functions of the field  $\rho$  (we could also have considered an arbitrary function  $F(\rho)$  rather than  $\rho$  as the coefficient of the  $(\nabla T)^2$  term but this can be absorbed by a field redefinition  $F(\rho) = \rho'$ ). Below we list the choices of these functions corresponding to the models reviewed in section (2.1)

$$\begin{array}{ll} \mathcal{L}_{TD} & A = B = \Lambda = \rho \\ \mathcal{L}_{ND} & A = 1, \rho = B = \Lambda = 0 \\ \mathcal{L}_{NRD} & A = B = 0, \Lambda = \rho \\ \mathcal{L}_{GD} & A = 0, B = 1, \Lambda = \rho \end{array} \quad (2.14)$$

Among these models we find many new ones such as  $A = 1, B = 0, \Lambda = \rho$  and of course all the models that one obtains by choosing  $A, B, \Lambda$  of a more general than linear form (notice however that unless  $A = 0$  we may absorb  $A$  into the  $W_j$  and have only  $B$  as a free function at our disposal. We keep both  $A, B$  so that we can treat the general case in a unified form).

## 2.5 General Hamiltonian Analysis of Theory Space

The general Hamiltonian analysis of the action corresponding to the sum of (2.13) and the standard matter and geometry contributions proceeds roughly as follows (we consider the case that  $\rho \neq 0$  and that not both of  $A, B$  vanish, more singular cases can be treated similarly, see [17] for all the details in the case  $A, B \neq 0$ ):

Computing the momenta  $P, P_j$  conjugate to  $T, S^j$ , the momenta  $\pi, \pi^j$  to  $\rho, W_j$  and the momenta



$\Pi, \Pi_a$  to lapse and shift functions  $N, N^a$  in the 3+1 split of the action, we discover the primary constraints  $z = \pi = 0$ ,  $z^j = \pi^j = 0$ ,  $Z = \Pi = 0$ ,  $Z_a = 0$  as well as another set of 3 (2) linearly independent constraints  $\zeta^j = 0$  for  $A \neq 0$  ( $A = 0$ ) which involves only  $P, P_j, W_j$  ( $P_j, W_j$ ), in both cases demand that  $P_j W_k - W_k P_j = 0$  and which come from the fact that one cannot solve for 3 (2) of the velocities  $\dot{S}^j$ . Altogether the canonical Hamiltonian depends on 11 (10) undetermined velocities. The stability analysis of the primary constraints with respect to the canonical Hamiltonian  $H_{\text{can}}$  yields that one can solve

$$\{\pi^j, H_{\text{can}}\} = 0, \quad j = 1, \dots, 3(2) \quad \{\zeta^j, H_{\text{can}}\} = 0, \quad j = 1, \dots, 3(2) \quad (2.15)$$

for 6 (4) of the velocities and that there are the 5 (6) secondary constraints

$$s = \frac{\partial H_{\text{can}}}{\partial \rho} = 0, \quad C^{\text{tot}} = \frac{\partial H_{\text{can}}}{\partial N} = 0, \quad C_a^{\text{tot}} = \frac{\partial H_{\text{can}}}{\partial N^a} = 0, \quad (K = \{\pi^3, \partial H_{\text{can}}\} = 0) \quad (2.16)$$

The secondary constraints can be stabilised by solving for 1 (2) of the remaining unfixed velocities so that altogether 7 of 11 (6 of 10) have been fixed, leaving in both cases  $\dot{N}$ ,  $\dot{N}^a$  undetermined. Altogether 5 (6) secondary constraints were found leading to altogether 16 constraints.

One finds that the 4 pairs  $(z, s), (z^j, \zeta^j)$ ;  $j = 1, \dots, 3$  ( $(z, s), (z^j, \zeta^j)$ ,  $j = 1, 2; (z^3, K)$ ) form a second class system while linear combinations of  $Z, Z_a, C^{\text{tot}}, C_a^{\text{tot}}$  with the second class constraints forms a first class set. The Dirac bracket between functions independent of  $\rho, W_j, N, N^a$  and their conjugate momenta reduces to the original Poisson bracket. This follows from the fact that such functions have vanishing Poisson brackets with the constraints  $z, z^j$  which mutually commute among themselves. The inverse of the Dirac matrix between the second class constraints is therefore such that the difference between the difference between the Poisson bracket and the Dirac bracket involves at least one Poisson bracket with one of the  $z, z^j$ . See section 4 or [17] for more details. One then solves the second class constraints strongly thereby eliminating  $\rho, W_j$  from the canonical Hamiltonian altogether. The result is the same as if one had eliminated right from the beginning  $\rho, W_j$  by using  $\partial H_{\text{can}}/\partial \rho = \partial H_{\text{can}}/\partial W_j = 0$  and dropping the primary constraints from the Hamiltonian. This follows because these four equations are just four of the second class constraints.

The canonical Hamiltonian is then a linear combination of the first class constraints  $C^{\text{tot}}, C_a^{\text{tot}}$  which besides geometry and standard matter degrees of freedom, which we denote collectively by  $(q, p)$ , now only depend on  $(T, P)$ ,  $(P_j, S^j)$  if not both  $A, B$  vanish and otherwise only on  $(T, P)$ . Crucially, since in the first case the Hamiltonian depended only on the combination  $W_j S_{,a}^j$  and since  $W_j \propto P_j$  on the constraint surface of the second class constraints, the canonical Hamiltonian eventually depends only on the combination  $P_j S_{,a}^j$  as far as the  $S^j$  dependent terms are concerned. This combination, however, is weakly equal to  $-(C_a + P T_{,a})$  (see below) where  $C_a$  is the contribution of geometry and standard matter to the spatial diffeomorphism constraint. It is for that reason (extended Brown – Kuchař mechanism) that this particular combination was chosen in the original Lagrangian. In both cases, the explicit dependence on  $T$  in the Hamiltonian constraint is through  $T_{,a}$  only and the mechanism just displayed makes sure that this remains true also when substituting  $P_j S_{,a}^j$ .

The remaining analysis of the system then proceeds as follows. It follows from the above that in the first case the constraints can be written in the equivalent form

$$C^{\text{tot}} = P + h(T, q, p), \quad C_a^{\text{tot}} = P T_{,a} + P_j S_{,a}^j + C_a(q, p) \quad (2.17)$$

while in the second one has

$$C^{\text{tot}} = P + \tilde{h}(T, q, p), \quad C_a^{\text{tot}} = P T_{,a} + C_a(q, p) \quad (2.18)$$

In the timelike dust model the dependence of  $h$  on  $T$  even drops out completely while in (2.18) the fact that  $\tilde{h}$  depends only on  $q^{ab}T_{,a}T_{,b} \approx q^{ab}C_aC_b/P^2$  can be used to further massage the constraints into the form

$$C^{\text{tot}} = P + h(q, p), \quad C_a^{\text{tot}} = PT_{,a} + C_a(q, p) \quad (2.19)$$

In all other models,  $h$  keeps an explicit dependence on  $T_{,a}$ . Nevertheless, in all models the Hamiltonian constraints in (2.17), (2.18) and (2.19) strongly Poisson commute. This is because they are first class by construction but since the Poisson bracket eliminates the dependence on  $P$ , the bracket must actually vanish identically in the four reference field case ( $\rho \neq 0$  and not both  $A, B$  vanish). Notice, however, that this only implies that also the  $h$  strongly Poisson commute if they are independent of  $T$ . In the single reference field it is more complicated to show that the  $h$  strongly commute as has been discussed in [10] and [6].

In the case of model (2.17) one can now perform a symplectic reduction of the spatial diffeomorphism constraint by pulling back all tensors and spinors by the diffeomorphism  $x \mapsto \sigma^j = S^j(x)$  which is a canonical transformation [1, 17] to the effect that all degrees of freedom except  $P_j, S^j$  remain canonical pairs when expressed in the new frame while in the new frame the momentum conjugate to  $S^j$  becomes  $([\partial S/\partial x]^{-1})^a_j C_a^{\text{tot}}$  so that this canonical pair drops out from our attention. The remaining degrees of freedom are spatially diffeomorphism invariant when expressed in this frame. We will denote them by  $(q', p', T', P')$  in order to distinguish them from  $(q, p, T, P)$ . In case of (2.19), spatial diffeomorphism invariant quantities must be constructed by other means. We will also pull back  $C^{\text{tot}}(x)$  which then becomes the same function  $C^{\text{tot}}(\sigma)$  of  $q', p', T', P'$  at  $\sigma$  as  $C^{\text{tot}}(x)$  was of  $q, p, T, P$  at  $x$ .

The final reduction of the system now employs the projector formula discovered in [11, 12]. See for a review in [17] in the notation used here. Let  $F', F$  be spatially diffeomorphism invariant functions on the phase space respectively which depend only on  $(q', p')$  in case of (2.17) and only on  $(q, p)$  in case of (2.19) respectively. Let

$$O'_{F'}(\tau) := (\exp(\{C^{\text{tot}'}[g'], \cdot\}) \cdot F')_{g'=T'-\tau}; \quad O_F(\tau) := (\exp(\{C^{\text{tot}}[g], \cdot\}) \cdot F)_{g=T-\tau}; \quad (2.20)$$

The notation means that we compute the Hamiltonian flow of the constraint indicated when smeared with a numerical test function  $g'$  or  $g$  and then set  $g' = T' - \tau$  or  $g = T - \tau$ . We cannot directly insert  $T - \tau$  into the exponent due to the phase space  $T$  dependence. Formula (2.20) is rather remarkable in several aspects. First of all (2.20) strongly Poisson commutes with all constraints. Secondly, we have the equal time  $\tau$  Poisson brackets

$$\{O'_{F'}(\tau), O'_{G'}(\tau)\} = O'_{\{F', G'\}^*}(\tau) \quad (2.21)$$

where  $\{., \cdot\}^*$  is the Dirac bracket with respect to the second class pair  $T, C^{\text{tot}'}$  (similar statements hold for the unprimed quantities). For quantities  $F', G'$  independent of  $P'$  it reduces to the normal Poisson bracket. It follows that if  $F', G'$  are conjugate so are  $O'_{F'}(\tau), O'_{G'}(\tau)$ . Secondly we have due to the mutual strong commutativity of the  $C^{\text{tot}'}(\sigma)$

$$\begin{aligned} \frac{d}{d\tau} O'_{F'}(\tau) &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int d^3\sigma_1 \dots \int d^3\sigma_n [\tau - T'(\sigma_1)] \dots [\tau - T'(\sigma_{n-1})] \{C^{\text{tot}}(\sigma_1), \{\dots\{C^{\text{tot}}(\sigma_n), F'\}\dots\}\} \\ &= O'_{\{C^{\text{tot}'}[1], F'\}}(\tau) = O'_{h'[1], F'}(\tau) = O'_{\{h'[1], F'\}^*}(\tau) \\ &= \{O'_{h'[1]}(\tau), O'_{F'}(\tau)\} \end{aligned} \quad (2.22)$$

where in the second step we realised the most inner Poisson bracket as the one between  $C^{\text{tot}'}[1] = \int d^3\sigma C^{\text{tot}'}(\sigma)$  and  $F'$ , in the third we used independence of  $F'$  on  $T'$  and that  $C^{\text{tot}'} = P' + h'$ ,

in the fourth we used independence of both  $h', F'$  of  $P'$  and in the last we used (2.21). The next property we need is that  $F' \mapsto O'_{F'}(\tau)$  preserves the multiplicative and additive structure on the Abelian algebra of phase space functions, that is,

$$O'_{F'}(\tau) = F'(O'_{q'}(\tau), O'_{p'}(\tau), O'_{T'}(\tau)) \quad (2.23)$$

if  $F' = F'(q', p', T')$  depends on  $q', p', T'$  only. Now, the observables associated to the time reference field is given by  $O'_{T'}(\tau) = \tau$  where  $\tau$  is a spatial constant. Likewise we obtain for the spatial derivative of  $T$  with respect to the (dust) spatial coordinates  $T_{,\sigma^j}$  the following observable  $O'_{T', \sigma^j}(\tau) = \partial_{\sigma^j} \tau = 0$ . Accordingly, since  $h'$  only depends on the derivatives of  $T'$  because we assumed that we do not consider potential terms of the reference matter fields in all models and not on  $T'$  itself we can define the intermediate Hamiltonian by

$$H' := \int d^3\sigma h'(T', q', p')_{T'=0} \quad (2.24)$$

Then (2.22) becomes

$$\frac{d}{d\tau} O'_{F'}(\tau) = \{O'_{H'}(\tau), O'_{F'}(\tau)\} = O'_{\{H', F'\}^*}(\tau) = O'_{\{H', F'\}}(\tau) \quad (2.25)$$

where independence of  $H'$  of  $P'$  was used again. The properties of  $F'$  that were used in this derivation is that  $F'$  is independent of both  $T', P'$ . Formula (2.25) therefore in particular applies to  $F' = H'$  whence

$$\frac{d}{d\tau} O'_{H'}(\tau) = 0 \quad (2.26)$$

is in fact  $\tau$  independent. It follows that (2.25) can be rewritten as

$$\frac{d}{d\tau} O'_{F'}(\tau) = \{\hat{H}, O'_{F'}(\tau)\} \quad (2.27)$$

where we have identified the physical, fully gauge invariant Hamiltonian

$$\hat{H} := O'_{H'}(0) = O'_{h'[1]}(0) \quad (2.28)$$

and in the second step we have made again use of  $O'_{T'}(0) = 0$ .

Alternatively, one can derive the fact that the physical Hamiltonian  $\hat{H}$  is not depending on time in the models considered as follows:

In case the function  $F'$  depends explicitly on  $\tau$  the formula in (2.22) needs to be modified and we obtain

$$\frac{d}{d\tau} O'_{F'}(\tau) = \{O'_{h'[1]}(\tau), O'_{F'}(\tau)\} + \frac{\partial O'_{F'}}{\partial \tau} \quad (2.29)$$

Now, let us consider the case that  $h'[1]$  depends on  $q', p', T'$  but not on  $P'$ , then we have

$$h'[1] = \int d^3\sigma h'(T', q', p') \quad (2.30)$$

Using that the application of the projector  $F' \mapsto O'_{F'}(\tau)$  preserves the multiplicative and additive structure on the Abelian algebra of phase space functions, we obtain

$$\hat{H} = O'_{h[1]}(\tau) = \int d^3\sigma h'(O'_{T'}, O'_{q'}, O'_{p'}) \quad (2.31)$$

Assuming that we do not consider potential terms of the reference fields, as it is done in all models considered in this paper, they occur only with spatial derivatives. Now the observable

associated to the time reference field is just given by  $O'_{T'}(\tau) = \tau$ . Since  $\tau$  does not depend on the spatial (dust) coordinates we further get  $O'_{T',\sigma^j}(\tau) = \tau_{,\sigma^j} = 0$ . Consequently, in this case the physical Hamiltonian is of the form

$$O'_{h[1]}(\tau) = \int d^3\sigma h'(O_{q'}(\tau), O_{p'}(\tau)) \quad (2.32)$$

and has no explicit  $\tau$ -dependence. Knowing this, we can immediately see that it is constant with respect to the  $\tau$ -evolution using (2.29)

$$\frac{d}{d\tau} O'_{h'[1]}(\tau) = \{O'_{h'[1]}(\tau), O'_{h'[1]}(\tau)\} + \frac{\partial O'_{h'[1]}}{\partial \tau} = 0 \quad (2.33)$$

Therefore we can use any value of  $\tau$  for the fully gauge invariant Hamiltonian  $\hat{H}$  and define

$$\hat{H} := O'_{h'[1]}(0) \quad (2.34)$$

Using (2.23) these are precisely the Hamiltonian equations of motion with respect to the Hamiltonian (2.34). Notice that unless  $h'_{|T'=0}$  is a Kuchař density,  $H'$  itself is not invariant under the Hamiltonian flow of generated by  $C^{\text{tot}'}$ , whence in this case the projector on  $H'$  in (2.24) is not trivial and  $H'$  must be replaced by (2.28). Also, unless  $h'_{|T'=0}$  is a Kuchař density, it will not be preserved by  $\hat{H}$  whence  $\hat{H}$  in general has only the three sets of local Noether charges  $C'_j$  which arises as the pull back of  $C_a$  by the diffeomorphism  $S^j$  (notice that  $h', C^{\text{tot}'}$  are a scalar densities of weight one in  $\sigma$  space and that  $\tau - T$  is a scalar so that (2.20) maps the diffeomorphism invariant quantity  $h'[1]$  to a diffeomorphism invariant quantity.).

We now reformulate in manifestly gauge invariant form. Consider the Dirac observables  $\hat{q} = O'_{q'}(0)$ ,  $\hat{p} = O'_{p'}(0)$ ,  $\hat{H} = O'_{h'[1]}(0) = h'[1](\hat{q}, \hat{p}) = H'(\hat{q}, \hat{p})$ , where we assume that  $h[1]$  includes only derivatives of the time reference field. Then (2.21) assures that Poisson brackets between functions of the hatted quantities can be computed by computing the brackets between the same functions of the primed quantities and then evaluated at the hatted quantities.

### 3 Non Rotational Dust

We carry out some of the details sketched in section (2.5) for the choice  $A = B = 0, \Lambda = \rho$  from a manifestly gauge invariant viewpoint and compare the results with [24]. Of course, almost everything we say here is already contained in [1, 21]. We will show that this model is contained as a subsector in the model [21], both classically and in the quantum theory and also explain why proposal for the quantum theory in [24] cannot describe the physical sector of the model.

In the case of non rotational dust the primary constraints are  $\pi = \Pi = \Pi_a = 0$  and the canonical Hamiltonian is given by

$$H_{\text{can}} = N^a C_a^{\text{tot}} + N \left( C + \frac{1}{2} \left( \frac{P^2}{\rho \sqrt{\det(q)}} + \rho \sqrt{\det(q)} [q^{ab} T_{,a} T_{,b} + 1] \right) \right) + v\pi + U\Pi + U^a \Pi_a \quad (3.1)$$

with undetermined velocities  $v, U, U^a$ . Stability of the primary constraints requires

$$\begin{aligned} C_a^{\text{tot}} &= P T_{,a} + C_a = 0 \\ C^{\text{tot}} &= C + \frac{1}{2} \left( \frac{P^2}{\rho \sqrt{\det(q)}} + \rho \sqrt{\det(q)} [q^{ab} T_{,a} T_{,b} + 1] \right) = 0 \\ s &= -\frac{P^2}{\rho^2 \sqrt{\det(q)}} + \sqrt{\det(q)} [q^{ab} T_{,a} T_{,b} + 1] = 0 \end{aligned} \quad (3.2)$$

Since  $\{\pi, s\} \neq 0$ , the constraints  $S, \pi$  form a second class pair and stability of  $s$  can be ensured by choosing  $v$ . We add to  $C_a^{\text{tot}}$  the term  $\pi \rho_{,a}$  proportional to the second class constraint  $\pi$ . Then this enlarged constraint generates spatial diffeomorphisms on all variables contained in  $C_a^{\text{tot}}$ ,  $C^{\text{tot}}$  and thus preserves these secondary constraints. The smeared  $C^{\text{tot}}$  Poisson commute to a smeared spatial diffeomorphism constraint according to the hypersurface deformation algebra because, up to the factors of  $\rho$ , the dust contribution  $C^D$  to the Hamiltonian constraints coincides with that of a massless Klein-Gordon field plus a cosmological constant term,  $\{C, C^D\}$  does not involve Poisson brackets between derivatives of fields and  $C^D, C^D$  lets its derivatives act on the smearing fields only so that the factors of  $\rho, 1/\rho$  in  $C^D$  cancel for the same reason as  $\sqrt{\det(q)}, 1/\sqrt{\det(q)}$ . It follows that there is only one second class pair. The corresponding Dirac bracket  $\{f, g\}^*$  differs from the Poisson bracket by terms that involves terms of the form  $\{\pi, f\}\{\pi, g\}$ ,  $\{s, f\}\{\pi, g\}$ ,  $\{\pi, f\}\{s, g\}$  since  $\{\pi, \pi\} = 0 \neq \{s, s\}, \{s, \pi\}$  (the inverse of the Dirac matrix  $\Delta_{ij} = \{s_i, s_j\}$ ,  $s_1 = s$ ,  $s_2 = \pi$  has the zeroes in complementary places as compared to the Dirac matrix itself). Hence, the Dirac bracket of phase space functions independent of  $\pi, \rho$  coincides with the Poisson bracket. We solve the second class constraints strongly and obtain

$$\pi = 0, \quad \rho^2 = \frac{P^2}{\det(q)[1 + q^{ab}T_{,a}T_{,b}]} \quad (3.3)$$

Choosing a sign  $\epsilon$  for  $\rho$  we find that the canonical Hamiltonian is a linear combination of first class constraints

$$\begin{aligned} C_a^{\text{tot}} &= PT_{,a} + C_a = 0 \\ C^{\text{tot}} &= C + \epsilon|P|\sqrt{1 + q^{ab}T_{,a}T_{,b}} \end{aligned} \quad (3.4)$$

This can be massaged into

$$C^{\text{tot}} = C + \epsilon\sqrt{P^2 + q^{ab}C_aC_b} \quad (3.5)$$

using the first equation in (3.4). On the constraint surface,  $C$  has the sign  $-\epsilon$  (unless all quantities vanish). Inverting for  $|P|$  we find

$$|P|^2 = C^2 - q^{ab}C_aC_b \quad (3.6)$$

which is constrained to be non negative. Thus, choosing a sign  $\delta$  for  $P$  we may replace (3.5) by

$$C^{\text{tot}} = P - \delta\sqrt{C^2 - q^{ab}C_aC_b} = P - \delta h \quad (3.7)$$

In the limit  $q^{ab}C_aC_b \rightarrow 0$  (3.5) and (3.7) become  $C + \epsilon|P| = P - \delta|C| = 0$  which is consistent for either choice of sign  $\epsilon, \delta$ . It is customary to choose  $\epsilon = 1 = -\delta$  (this is one of the four possible subsectors of phase space that one must pick in these local considerations). As  $H$  no longer depends on  $T$  it must be a Kuchař density which maybe checked explicitly. We stress here that that in this model, as one can see explicitly from (3.4) or (3.5), the sign of  $C$  is constrained, specifically  $-\epsilon$  is the sign of  $C$ . In particular, for  $\epsilon = 1$  (positive kinetic energy term),  $C$  is constrained to be negative.

The physical Hamiltonian becomes by the general considerations of the previous section the manifestly positive expression (whatever choice of  $\epsilon, \delta$  would have been made, (3.8) has positive sign)

$$\hat{H} = \int d^3x h(x) \quad (3.8)$$

generating time evolution of functions of the form

$$O_F(\tau) = [\exp(\{C^{\text{tot}}[g], \cdot\}) \cdot F]_{g=T-\tau} \quad (3.9)$$

where  $F$  is spatially diffeomorphism invariant and independent of  $T, P$ . The functions (3.8) and (3.9) are full Dirac observables of the theory and their algebra may be computed on the constraint surface  $C^{\text{tot}} = C^{\text{tot}} = 0$ . Since they are gauge invariant, their algebra may be evaluated on the gauge cut  $T = \tau_0$  for some fixed  $\tau_0 = \text{const.}$ , say  $\tau_0 = 0$  within that constraint surface. On that gauge cut we have  $C_a^{\text{tot}} = C_a = 0$  and  $h = |C|$  as well as  $O_F(\tau) = \exp([\tau - \tau_0]\{h[1], \cdot\}) \cdot F$ , in particular  $O_F(\tau_0) = F$ . The generator of time evolution is evidently still given by  $H = h[1]$  in (3.8) but on the gauge cut its Hamiltonian vector field reduced to that of  $|C|[1]$  as follows from

$$\begin{aligned} \{H, f\} &= \int d^3x \frac{1}{2h} (2C\{C, f\} - 2q^{ab}C_b\{C_a, f\} - C_a C_b \{q^{ab}, f\})_{C_a=0} \\ &= \int d^3x \text{sgn}(C) \{C, f\} = \{|C|[1], f\} \end{aligned} \quad (3.10)$$

A different way to see this is to start from the formulation (3.4) and to write the Hamiltonian constraint in the equivalent form

$$C^{\text{tot}} = P + \frac{|C|}{\sqrt{1 + q^{ab}T_{,a}T_{,b}}} =: P + \tilde{h} \quad (3.11)$$

where the sign choice  $\epsilon = 1 = -\delta$  has been adopted. The constraints (3.11) are still Abelian and the machinery sketched in the previous section applies but  $\tilde{h}$  is no Kuchař density any longer and  $\tilde{h}[1]$  is no longer a Dirac observable. However, the theory sketched there shows that the physical Hamiltonian is given by (notice that  $\tilde{O}(\tau)$  is now defined in terms of  $\tilde{h}$  while  $O(\tau)$  is defined in terms of  $h$ )

$$\hat{H} = \tilde{O}_{\tilde{h}[1]}(0) = \tilde{O}_{|C|[1]}(0) \quad (3.12)$$

since  $O_{T,a}(0) = 0$ . As the algebra of the  $F$  and  $|C|$  is isomorphic to the algebra of the  $O_F(0)$  and  $O_{|C|}(0)$  we arrive at the same picture at this fully gauge invariant level as in its gauge fixed version. Yet a third way to see this is to start from (3.8) and to make use of the fact that  $h$  is a Kuchař density. Then trivially  $H = O_H(0) = \hat{H}$ . On the other hand in  $O_H(0)$  we may replace  $C_a$  appearing in  $h = \sqrt{C^2 - q^{ab}C_a C_b}$  by  $C_a^{\text{tot}}$  as  $O_{T,a}(0) = 0$ . Writing  $\hat{C} = O_C(0)$ ,  $\hat{C}_a^{\text{tot}} = O_{C_a^{\text{tot}}}(0)$  etc. we have

$$\hat{H} = \int d^3x \sqrt{[\hat{C}]^2 - \hat{q}^{ab}\hat{C}_a^{\text{tot}}\hat{C}_b^{\text{tot}}} \quad (3.13)$$

and now a calculation similar to (3.10) reveals that on the constraint surface we may replace  $\hat{H}$  by  $O_{|C|[1]}(0)$ . We conclude that on the constraint surface all three approaches are equivalent.

Notice that without the additional fields  $S^j$  spatially diffeomorphism functions  $F$  are difficult to construct: Typically one will form them from scalar densities of weight one, built from the fields  $q, p$  and their spatial covariant derivatives. These are no longer elementary functions such as  $q, p$  in terms of which a quantisation is typically easy. The equal time Poisson algebra of the observables  $O_F(\tau)$  is isomorphic to that of the  $F$  since they do not depend on the  $P, T$ . However, since it is unknown how to construct a complete and independent set of such  $F$  and thus how to write their algebra in closed form, it is not possible to use the  $O_F(0)$  or equivalently the  $F$  as a platform for quantisation and thus a full reduced phase space quantisation is impossible for practical reasons. One thus has to resort to a hybrid scheme such as in [22] where one starts from the phase space unreduced with respect to the spatial diffeomorphism constraint and quantises the  $O_f(0)$  whose algebra is isomorphic<sup>3</sup> to that of the  $f$ . The spatial diffeomorphism constraint is then solved in the quantum theory following [31]. However, while

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<sup>3</sup>In [22] actually the  $f$  were quantised and the projector formula  $O(\tau)$  was implemented in the quantum theory. The results are equivalent.

this equips the theory with a physical Hamiltonian  $H$  (for the matter model considered in [5, 6] rather than that model of non rotational dust considered as a special case of [1] used in this section), the task of implementing the  $O_F(0)$  remains to be performed and the reservations of [23] are sustained.

Coming back to the model of this section that was studied in [24] the spatial diffeomorphism constraint needs to be solved at the quantum level for the reason that likewise to [22] only one (time) reference field is involved in that model. Hence in order to describe the physical sector of the quantum theory, the same procedure as in [22] must be adopted as far as the quantum theory is concerned and the physical Hamiltonian needs to be quantised on the diffeomorphism invariant Hilbert space. However, the authors of [24] suggest a different procedure. Rather than quantising the theory and the physical Hamiltonian  $|C|$  on the diffeomorphism invariant Hilbert  $\mathcal{H}_{\text{diff}}$  space constructed in [31] and employed in [22], they suggest to use the algebraic quantum gravity framework of [21] and to quantise  $q, p, |C|$  directly on the corresponding Hilbert space  $\mathcal{H}_{\text{AQG}}$ . Since that Hilbert space is based on abstract rather than embedded graphs, their viewpoint seems to be that therefore the spatial diffeomorphism constraint (and by the above gauge fixing viewpoint also the Hamiltonian constraint) can be considered as solved so that  $\mathcal{H}_{\text{AQG}}$  is actually the physical Hilbert space with the simplest possible physical Hamiltonian  $C$  on it. We make five comments about this:

1. First of all, since really the physical Hamiltonian is  $|C|$  rather than  $C$  there is secretly still a square root  $|C| := \sqrt{C^2}$  involved which makes  $|C|$  no better than  $h = \sqrt{C^2 - q^{ab}C_a C_b}$  when quantising the theory. However, note that there is no principal problem with the square root because one can use the methods of [26] to handle it. The authors of [24] could of course take the point of view that they fix the sign of  $C$  classically and then quantise the corresponding part of the phase space in order to avoid the sign function at the quantum level. However, this is almost impossible to control in practical terms, since it requires detailed knowledge about the spectrum of  $C$ . The only practical way of ensuring the sign is to introduce the absolute value (that is a square root). Alternatively, the authors of [24] could argue to relax the typical energy conditions of classical General Relativity which however is not unproblematic. In any case, this discussion shows, that the claim that the physical Hamiltonian simplifies in the maximal sense and just reduces to  $C$  deserves further discussion. By contrast, as we will discuss in the next section, there exists already a model [3], the Gaussian dust model, in which one can obtain physical Hamiltonian densities that simplify in the maximal sense and are just given by  $C$  without the need to fix signs at the classical level.

2. As noticed previously in the literature (see e.g. [27] and references therein) the Hilbert space  $\mathcal{H}_{\text{AQG}}$  cannot be identified with the physical Hilbert space for models where the spatial diffeomorphism constraint is not solved already classically. We repeat the argument here for completeness:

The physical interpretation of spin network functions in  $\mathcal{H}_{\text{AQG}}$  whose abstract graphs are finite subgraphs of an infinite abstract graph is obtained by embedding them into the given spatial manifold  $\sigma$  under consideration. Any such embedding  $Y$  establishes an isomorphism, at least when  $\sigma$  is compact, with a subspace of the kinematical Hilbert space  $\mathcal{H}_{\text{LQG}}$  of LQG. In that Hilbert space, spin network functions based on diffeomorphic graphs are gauge equivalent. Hence must be their preimages in  $\mathcal{H}_{\text{AQG}}$  under  $Y$ . For instance, if the infinite algebraic graph is cubic then all Wilson loop functions around plaquette loops should be identified.

The physical sector of the quantum theory in AQG is determined by the kernel of the extended master constraint, which includes, for the reasons mentioned above, also a master

constraint version of the spatial diffeomorphism constraint at the algebraic level. The quantisation of the (not necessarily algebraic graph-preserving) extended master constraint is presented in [26] using techniques from [25]. (Note that the continuum version of the spatial diffeomorphisms constraint used at the embedded level has no counterpart at the algebraic level). Therefore, when solving the Hamiltonian constraint classically, as it is done in [24], one still needs to find the solutions of the operator corresponding to the classical expression

$$M := \int d^3\sigma \frac{q^{ab}C_a C_b}{\sqrt{\det(q)}} \quad (3.14)$$

in order to solve the spatial diffeomorphism constraint at the quantum level and thus describe the physical sector of the theory. The quantisation of such an operator was presented in [25, 26], however in the case of the model in [24] this quantisation cannot be copied. The reason is the following: Assume that one is able to find the solution space of the (corresponding) operator in (3.14). Then this solutions space is not left invariant under the action of the physical Hamiltonian because  $q^{ab}C_a C_b$  will not commute with the physical Hamiltonian densities given by  $C$  (or  $|C|$  respectively when the sign is not fixed at the classical level). Therefore one needs to find a quantisation of  $M$  such that the corresponding operator commutes with  $C$  modulo  $M$ . Such a quantisation<sup>4</sup> so far does not exist in the literature and also has not been worked out in [24].

3. The authors of [24] sketch the prospects that their model may have. The list of corresponding items is almost identical to that of [21] which predates [24]. Hence let us compare the models of [24] and [21]. The model of [21] is based on the general model in [1] rather than the special case of non rotational dust also discussed in [1]. It thus performs, in contrast to [24], a complete reduction of the physical phase space since it contains also the scalar fields  $S^j$ . The authors of [21] outline a quantisation of the algebra of full quantum observables and the physical Hamiltonian  $H$ , which is isomorphic to the simple algebra of the  $q, p$ , on both Hilbert spaces  $\mathcal{H}_{\text{LQG}}$  and  $\mathcal{H}_{\text{AQG}}$  which rightfully can be called the *physical Hilbert space* in contrast to the model [24]. The price to pay is that the physical Hamiltonian is now given by the quantisation of the term

$$H = \int d^3x \sqrt{|C^2 - q^{ab}C_a C_b|} \quad (3.15)$$

The additional absolute value respects the fact that the quantity under the square root is constrained to be non negative in the classical theory and in this form (3.15) allows for a well defined quantisation as discussed in [21]. Since the spatial diffeomorphism constraint reads

$$C_a^{\text{tot}} = PT_{,a} + P_j S^j_a + C_a \quad (3.16)$$

even in the gauge  $T = \tau$  and  $S^j = \sigma^j$  (3.15) does not reduce to  $|C|[1]$ . This is because in this gauge (3.16) becomes

$$C_a^{\text{tot}} = P_j \delta^j_a + C_a \quad (3.17)$$

However, this is neither a problem, since semiclassical tools are available in order to deal with square root Hamiltonians [26], nor worse than working with  $|C|$ .

4. It is easy to see that the model [24] corresponds to a subsector of the model [21]. This can be demonstrated both at the classical and the quantum level.

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<sup>4</sup>Note that it is not enough to commute modulo  $C_a$  which classically is trivially the case because  $C_a$  in contrast to  $M$  is not a well defined operator.



At the classical level this corresponds to the observation that the portion of phase space where  $C_a = 0$  is left invariant by (3.15). Let us discuss the degrees of freedoms in both models. In [24] one has gravity plus one scalar field and in [21] gravity plus four scalar fields. In both models the total spatial diffeomorphism constraint is required to vanish yielding at the level of the gauge  $T = \tau$  for the model [24] the constraint  $C_a = 0$ . When we consider for the model in [21] the gauge  $T = \tau$  and  $S^j = \sigma^j$  we obtain  $C_a + P_j \delta_a^j = 0$  where  $P_j$  are the momenta conjugate to  $S^j$ . Let us define  $P_a := P_j \delta_a^j$  then in the model [21] we have  $C_a = -P_a$  and it can be shown that  $P_a$  is a constant of motion and thus so is  $C_a$ . In particular, requiring  $C_a$  to vanish in the model of [21] means  $P_a = 0$  yielding 3 additional constraints reducing the number of degrees of freedom to that of the model [24]. Further evidence is obtained by noting that  $P_a = 0$  means  $P_j = 0$  and from this follows, using  $W_j \sim P_j$ , that the Lagrange multiplier fields  $W_j$  are constrained to vanish. This means that the reference fields  $S^j$  are no longer present in the action. Hence we arrive at the special case of non-rotating dust, demonstrating the embedding at the classical level.

At the quantum level, the embedding is less intuitive but nevertheless can be shown. Let us first discuss the case where we choose the version of [21] where one quantises on  $\mathcal{H}_{\text{LQG}}$ . One observes that the physical Hamiltonian operator  $H$  is invariant under diffeomorphism on the dust manifold  $S(\sigma)$  and thus commutes with its generator  $C_a$  where the index  $a$  here needs to be understood as labelling coordinates on the dust manifold  $S(\sigma)$ . One would rather call  $C_a$  a generator of symmetries in dust space than a generator of gauge transformations. This symmetry generator is implemented on the physical Hilbert space which is the standard kinematical Hilbert space  $\mathcal{H}_{\text{LQG}}$  of LQG. Now given the physical Hilbert space, one can look for functions that are invariant under the symmetry transformations generated by  $C_a$  and those (dust) diffeomorphism invariant 'functions', that are rather distributions, will be the 'subset' of functions that are additionally annihilated by  $C_a$  and hence also by the operator  $q^{ab}C_aC_b$  when  $q^{ab}C_aC_b$  is properly quantised as to annihilate diffeomorphism invariant distributions. On those diffeomorphism invariant distributions  $H$  reduces to  $|C|$ .

Now, let us consider the second possibility where one chooses the version of [21] that is quantised on  $\mathcal{H}_{\text{AQG}}$  and see how the quantum theory of [24] is embedded in that of [21]. Suppose we have fully reduced the Hamiltonian and spatial diffeomorphism constraint at the classical level, then the operator associated with the classical expression shown in (3.14) is a generator of symmetries rather than gauge transformations at the physical Hilbert space  $\mathcal{H}_{\text{AQG}}$ . Likewise to the standard LQG quantisation, one can now look for 'functions' (distributions) that are annihilated by that operator and consider the solution space as a 'subspace' of the physical sector of the theory. However, here in general this 'subspace' will not be left invariant under the dynamics unless, as discussed before the operators  $M, C$  (3.14) are quantised in such a way that they mutually commute. This would in any case be a desirable feature of the theory but this is not granted by the naive quantisations presented in [26] and employed in [24]. In other words, whenever  $M$  is defined in a satisfactory way in the model [24] at the algebraic level then it corresponds to a subsector to the algebraic version of [21].

Finally, we observe the following curiosity when working in the standard LQG framework. Let us consider the 'subspace' of functions that are invariant under the symmetry transformation generated by the  $C_a$  on  $\mathcal{H}_{\text{LQG}}$ . Now this 'subspace' has mathematically the same structure as the standard diffeomorphism invariant Hilbert space of LQG. Therefore, on this 'subspace' the physical Hamiltonian reduces to an expression where all terms involving  $q^{ab}C_aC_b$  can be neglected and furthermore a graph-modifying quantisation could be applied. Hence, in this sense, in this subsector knot class modifying models can be redis-

covered although it would be a bit artificial to force a knot class modifying quantisation in that particular 'subspace' whereas on the 'rest' of the Hilbert space a graph preserving quantisation is adopted.

5. A final comment concerns the claim made in [24] that their physical Hamiltonian is quantised free of "anomalies" as compared to [26]. Just to avoid confusion, let us try to interpret this statement. If what is meant that one single Hamiltonian operator (as compared to an infinite number of Hamiltonian constraint operators) commutes with itself, the statement is empty. If what is meant is that the Hamiltonian densities (as compared to the Hamiltonian which is the integral of the densities) mutually commute which is equivalent to the mutual commuting of the Hamiltonian constraint operators, the statement is wrong at the algebraic level in which the authors of [24] are working. The commutator of two Hamiltonian densities is not an algebraic version of the diffeomorphism constraint and in that sense there is an anomaly. This is again the reason why in [26] a master constraint approach towards all constraints has been adopted. If what is meant is that within the embedded LQG framework the Hamiltonian constraints commute on diffeomorphism invariant contributions, this is correct when using the quantisation of [32] but not when using the algebraic quantisation of [26] that the authors copy. In any case the quantisation of [32] is inappropriate for a physical Hamiltonian rather than an infinite number of Hamiltonian constraints because the resulting Hamiltonian would not even be symmetric in that quantisation scheme (for constraints, symmetry is not necessary as one is only interested in the joint kernel). Rather, as pointed out in [21], one must use the graph preserving quantisation scheme (and a symmetric ordering) in order to obtain a symmetric operator which is spatially diffeomorphism invariant. Finally, as already pointed out in [21], the issue of anomalies is much less critical in this reduced phase space quantisation approach as compared to the operator constraint quantisation approach because the number of degrees of freedom has been correctly reduced already at the classical level. In that sense there can be no anomaly. However, one might still be interested in implementing the classical hypersurface deformation algebra as a physical principle to reduce the quantisation ambiguities and this remains true for all models considered so far.

## 4 Gaussian Dust

It transpires that an optimal model would be such that 1. the physical Hilbert space is the usual  $\mathcal{H}_{\text{LQG}}$  (or  $\mathcal{H}_{\text{AQG}}$ ) and 2. the physical Hamiltonian density is equivalent to just  $C$  and not  $|C|$  or  $\sqrt{|C^2 - q^{ab}C_a C_b|}$ . None of the models [21, 22, 24] has both features. Remarkably, we find exactly such a model in [3]. In what follows we sketch the classical treatment of this model in some detail since it differs slightly from that of [17, 18]. In contrast to [3] we will not simply eliminate the Lagrange multiplier fields by their equations of motion but rather go through a careful Dirac treatment as outlined in section 2.5. The quantisation of this model will be copied from [21, 32] and we will therefore be brief on that point. For the physical interpretation of this model as a Gaussian reference fluid we refer to [3]. We just mention here that variation of the action with respect to  $\rho, W_j$  respectively leads to the conditions  $U^\mu U_\mu = -1$ ,  $U_\mu V_j^\mu = 0$  on the vectors  $U^\mu = g^{\mu\nu} T_{,\nu}$ ,  $V_j^\mu := g^{\mu\nu} S_{,\nu}^j$  and the equations of motion for  $T, S^j$  yield the geodesic equation  $\nabla_U U = 0$  and the conservation equation  $\nabla_\mu (W_j U^\mu) = 0$ . Hence the integral curves of  $U$ , labelled by  $S^j(x) = \sigma_j = \text{const.}$  describe an observer in geodesic motion and the vectors  $V_j$  are orthogonal to the corresponding  $T = \text{const.}$  surfaces. Hence in the corresponding reference frame the metric assumes the Gaussian form with unit lapse and zero shift. The energy momentum tensor turns out to be  $T^{\mu\nu} = \rho U^\mu U^\nu + \sum_j W_j V_j^{(\mu} U^{\nu)}$  which has no spatial trace with  $U_\mu U_\nu + g_{\mu\nu}$  and thus no pressure whence the name "dust" is appropriate.

The Gaußian dust action reads explicitly

$$\mathcal{L}_{\text{GD}} = -|\det(g)|^{1/2} \left\{ \rho \frac{1}{2} [g^{\mu\nu} T_{,\mu} T_{,\nu} + 1] + g^{\mu\nu} T_{,\mu} [W_j S^j_{,\nu}] \right\} \quad (4.1)$$

Performing the 3+1 split in the ADM frame we obtain

$$\begin{aligned} g^{\mu\nu} T_{,\mu} T_{,\nu} &= -[L_n T]^2 + q^{ab} T_{,a} T_{,b}, \quad g^{\mu\nu} T_{,\mu} [W_j S^j_{,\nu}] = -[L_n T] [W_j L_n S^j] + q^{ab} T_{,a} [W_j S^j_{,b}]; \\ L_n = n &= \frac{1}{N} [\partial_t - N^a \partial_a] \end{aligned} \quad (4.2)$$

where  $N, N^a$  are the usual lapse and shift functions and  $q_{ab}$  is the intrinsic metric on the 3-manifold  $\sigma$  (with inverse  $q^{ab}$ ) which is mapped via a one parameter family of embeddings  $Y_t$  into into a one parameter family of spacelike hypersurfaces  $\Sigma_t = Y_t(\sigma)$  that foliate  $M$ . The timelike vector field  $n$  is unit normal to the foliation and its action on the scalars in (4.2) coincides with the Lie derivative  $L_n$ .

Performing the Legendre transform we find together with  $\sqrt{|\det(g)|} = N\sqrt{\det(q)}$

$$\begin{aligned} P &:= \frac{\mathcal{L}_{\text{GD}}}{\partial \dot{T}} = \sqrt{\det(q)} \{ \rho [L_n T] + W_j [L_n S^j] \} \\ P_j &:= \frac{\mathcal{L}_{\text{GD}}}{\partial \dot{S}^j} = \sqrt{\det(q)} W_j [L_n T] \\ \pi &:= \frac{\mathcal{L}_{\text{GD}}}{\partial \dot{\rho}} = 0 \\ \pi^j &:= \frac{\mathcal{L}_{\text{GD}}}{\partial \dot{W}_j} = 0 \end{aligned} \quad (4.3)$$

The detailed constraint analysis can be found in the appendix. From it follows that we have 8 first class constraints  $Z, Z_a, C^{\text{tot}}, C_a^{\text{tot}}$  and 8 second class constraints  $z, z_j, \zeta_I, s, K$ . Fortunately it is not necessary to compute the corresponding Dirac bracket explicitly by the following argument:

We arrange the second class constraints into 2 sets  $\{K_\mu^{(1)}(x)\}_{I=1}^4 = \{z, z^j\}(x)$  and  $\{K_\mu^{(2)}(x)\}_{I=1}^4 = \{\zeta_I, s, K\}(x)$ . Then the Dirac matrix

$$\Delta_{\mu\nu}^{IJ}(x, y) := \{K_\mu^{(I)}(x), K_\nu^{(J)}(y)\} \quad (4.4)$$

and its inverse assumes a square block structure of the form

$$\Delta = \begin{pmatrix} \Delta^{11} = 0 & \Delta^{12} \\ -[\Delta^{12}]^T & \Delta^{22} \end{pmatrix} \Rightarrow \Delta^{-1} = \begin{pmatrix} ([\Delta^{12}]^{-1})^T \Delta^{22} [\Delta^{12}]^{-1} & -([\Delta^{12}]^{-1})^T \\ [\Delta^{12}]^{-1} & 0 \end{pmatrix} \quad (4.5)$$

As we will eventually solve the second class constraints for the Lagrange multiplier fields  $\pi, \pi^j, \rho, W_j$  we are interested only in the restriction of the Dirac bracket to functions  $f$  of the variables  $q, p; T, P; S^j, P_j$ . For such functions we have  $\{K_\mu^1(x), f\} = 0$ . Since the difference between the Dirac bracket and the Poisson bracket between  $f, f'$  contains, due to (4.5), only terms with at least one of  $\{K_\mu^1(x), f\}$  or  $\{K_\mu^1(x), f'\}$ , the Dirac bracket and the Poisson bracket actually coincide on the functions of interest.

The solution of the second class constraints  $\zeta_I = K = s = 0$  is given by

$$\begin{aligned} W_I &= W_3 \frac{P_I}{P_3} \\ W_3^2 &= \frac{P_3^2}{Q^2(1 + q^{ab} T_{,a} T_{,b})} \\ \rho &= \frac{1}{P_3} \left( P - \frac{q^{ab} T_{,a} P_j S^j_{,b}}{1 + q^{ab} T_{,a} T_{,b}} \right) \end{aligned} \quad (4.6)$$

The solutions for  $W_I, \rho$  are not explicitly needed but we need to choose a sign  $\epsilon$  for  $W_3/P_3$ , insert the root of the second equation in (4.6) into the Hamiltonian constraint and see that the term proportional to  $\rho$  vanishes identically (since it enters linearly into the Hamiltonian constraint)

$$C^{\text{tot}} = C + \epsilon \left( P \sqrt{1 + q^{ab} T_{,a} T_{,b}} + \frac{q^{ab} T_{,a} [P_j S^j_{,b}]}{\sqrt{1 + q^{ab} T_{,a} T_{,b}}} \right) \quad (4.7)$$

while the spatial diffeomorphism constraint becomes

$$C_a^{\text{tot}} = P T_{,a} + P_j S^j_{,a} + C_a \quad (4.8)$$

Substituting for  $P_j S^j_{,a}$  in (4.7) we find the equivalent and simple form

$$C^{\text{tot}} = C + \epsilon \frac{P - q^{ab} T_{,a} C_b}{\sqrt{1 + q^{ab} T_{,a} T_{,b}}} \quad (4.9)$$

or equivalently

$$C^{\text{tot}} = P + h =: P + \epsilon C \sqrt{1 + q^{ab} T_{,a} T_{,b}} - q^{ab} T_{,a} C_b \quad (4.10)$$

We see that  $h$  is of the type described in section (2.5) so that the general theory outlined there applies. Moreover, we see a crucial difference with the model described in section 3: The sign  $\epsilon$  of  $W_3/P_3$  is unrelated to the sign of  $P, C$ . The choice of  $\epsilon$  in the Lagrangian also has no physical significance since the  $W_j$  dependent term is neither bounded from above nor from below. For definiteness we simply choose the phase space such that  $\epsilon = +1$ . Hence there is no absolute value of  $C$  involved in (4.10) and the physical Hamiltonian becomes simply, following the exposition of section 2.5

$$\hat{H} = \int_{\mathcal{S}} d^3\sigma \hat{C}(\sigma), \quad \hat{C}(\sigma) = C(\hat{q}, \hat{p}) \quad (4.11)$$

where  $\mathcal{S} = S(\sigma)$  is the dust particle manifold and  $\hat{q} = O'_{q'}(0)$ ,  $\hat{p} = O'_{p'}(0)$ , that is, it is the same function of  $\hat{q}, \hat{p}$  as is  $C$  of  $q, p$ . Here  $q', p'$  denotes the pull back of the fields  $q, p$  by the diffeomorphism  $\sigma^j = S^j(x)$  and  $O'(0)$  is the projector map defined in section 2.5. The Poisson algebras of the  $q, p$  and of the  $q', p'$  are identical. Notice that (4.11) is invariant under  $\text{Diff}(\mathcal{S})$ .

The quantisation of the system can now be copied from [32, 21]. If we work on the Hilbert space  $\mathcal{H}_{\text{LQG}}$  and want the quantum operator to have the symmetries of its classical counterpart then  $\hat{H}$  must preserve the subspaces  $\mathcal{H}_{\text{LQG}, \gamma}$  defined by the closed linear span of  $\text{SU}(2)$  invariant spin network functions over the graphs  $\gamma$  embedded in  $\mathcal{S}$  in order to be densely defined. In [21] it is described how this can be achieved using the notion of a minimal loop attachment and a corresponding projection operator onto  $\mathcal{H}_{\text{LQG}, \gamma}$ . In the framework of  $\mathcal{H}_{\text{AQG}}$  the operator  $\hat{H}$  does not need to preserve any of the subgraphs of the algebraic graph and can be defined in terms of next neighbour loops [21, 26] just as in lattice gauge theory. In both approaches a symmetric ordering of the Hamiltonian densities must be chosen. Here the following subtlety arises:

When considering  $|C|$  rather than  $C$  the ordering of  $C$  is chosen in such a way that  $C$  acts only at the vertices of a graph. This ordering is not symmetric but this does not matter because one considers the positive operator valued density  $|C| := \sqrt{C^* C}$ . However, when quantising  $C$  itself, such an ordering is not available. To make sure that  $C$  is symmetric and is densely defined (the danger being that the curvature term involved in  $C$  acts everywhere, not only at the vertices of

a spin network function) one uses the tools developed in [32] and writes

$$\begin{aligned}
1 &= \frac{(\det(e(x)))^2}{\sqrt{\det(q(x))}^2} = \left( \frac{\det(\{A(x), V_\epsilon(x)\})}{\sqrt{\det(q(x))}} \right)^2 \\
&= \lim_{\epsilon \rightarrow 0} \left( \frac{\det(\text{Tr}(A_\epsilon(x)^{-1} \{A_\epsilon(x), V_\epsilon(x)\}))}{V_\epsilon(x)} \right)^2 \\
&= \left( \frac{3}{2} \right)^6 \lim_{\epsilon \rightarrow 0} \left[ \det \left( (\text{Tr}(A_\epsilon(x)^{-1} \{A_\epsilon(x), V_\epsilon(x)^{2/3}\})) \right) \right]^2
\end{aligned} \tag{4.12}$$

where  $e(x)$  is the cotriad,  $A(x)$  is the connection,  $V_\epsilon(x)$  is the volume function of a neighbourhood of  $x$  with coordinate volume  $\epsilon^3$  and  $A_\epsilon(x)$  is a set of three  $\text{SU}(2)$  valued holonomy functions along three edges starting at  $x$  with linearly independent tangents at  $x$  and which span a coordinate volume  $\epsilon^3$  as well. As shown in [32], each of the two factors in (4.11) admits a well defined quantisation in the limit of vanishing regulator and acts only at vertices. We can therefore freely position one factor each from (4.11) to the outmost left and right of the operator and then order symmetrically (including the projections on  $\mathcal{H}_{\text{LQG},\gamma}$  in the case of a quantisation on  $\mathcal{H}_{\text{LQG}}$ ).

We notice that since  $C$  is not a Kuchař density, the physical Hamiltonian has less symmetries than for the model described in [17, 21], it does not Poisson commute with its density and thus it only has the Noether densities  $\hat{C}_j(\sigma)$  as conserved charges. However,  $\hat{C}(\sigma)$  becomes a Noether charge when the other three Noether charges  $\hat{C}_j(\sigma)$  vanish because the  $\hat{C}(\sigma)$ ,  $\hat{C}_j(\sigma)$  obey the hypersurface deformation algebra. Thus, on this sector of the classical theory, the results of [17, 18, 33] continue to hold which in the case of [17, 18] only assumed  $\hat{C}(\sigma)$  to be preserved and non vanishing while  $\hat{C}_j(\sigma)$  was kept arbitrary and in the case of [33] one was working with  $\hat{C}_j(\sigma) = 0$  anyway. As a result, the model described in this section is in agreement with the usual gravitational waves, cosmology, cosmological perturbation theory and black holes description as as described by geodesic test observers, following the analysis of [17, 18, 21].

## 5 Summary and Outlook

In this paper we have accomplished three tasks:

1. We have described all matter models considered so far for the purpose of deparametrisation of General Relativity with an eye towards Quantum Gravity starting from a general Lagrangian. This serves to bring order into the space of models studied already and those that can be within the theory space described. We have described the Hamiltonian analysis of these models in a uniform fashion.
2. Basically two types of models have been constructed: II. Those which deparametrise only time and I. those which deparametrise spacetime.

We have shown that those of type II. necessarily must implement spatial diffeomorphism invariance in the quantum theory. This is a non trivial subject for several reasons:

First, even though the unitary implementation of spatial diffeomorphisms in LQG is no problem, there is no unique Hilbert space structure on the diffeomorphism invariant distributions [31]. To fix the ambiguities, the implementation of the  $*$ relations among spatially diffeomorphism invariant observables may be consulted as a guiding principle, however, neither is a complete and algebraically independent set of such objects known even at the classical level nor is it clear that these can be quantised as to yield an honest representation of their  $*$ algebra (see [34] for the tremendous difficulties encountered already in the

much simpler setting of the closed bosonic string). This is due to the fact that spatially diffeomorphism invariant functions on phase space are typically not simple polynomials of the elementary fields. The considerations made in [23] indicate that such anomalies are a conceivable possibility and that no choice of the ambiguity parameters may exist for which the representation problem can be solved.

These difficulties are naturally avoided in the type I models where also the spatial diffeomorphism symmetry is reduced already at the classical level. The non trivial observation is that this can be done while keeping a very simple Poisson  $\ast$ algebra of basic fields so that one can easily find Hilbert space representations thereof. This also extends to the reduction of the symmetries generated by the Hamiltonian constraint, at least for the matter models considered here. This is quite remarkable when recalling the difficulties that one meets when trying to implement the Hamiltonian operator constraint in LQG, see e.g. the discussion in [21, 26, 28, 35].

3. The authors of [24] point out that their model, which belongs to type II, avoids a square root in the physical Hamiltonian density in contrast to the models considered in [21, 22]. We have shown that this statement has to be made more precise. If one insists on the usual energy conditions of classical General Relativity then the analysis shows that rather  $|C| = \sqrt{C^2}$  is the physical Hamiltonian density which again needs a square root. We remark that square roots are a nuisance rather than a caveat as semiclassical tools are available that can deal with this problem [26]. However, it is certainly true that the analysis simplifies significantly when there is no square root. Unfortunately, in type II models the quantum reduction of the spatial diffeomorphism constraint still poses a challenge. In view of what was just said, this leads to the natural question whether there are matter models within type I which lead to a physical Hamiltonian without square root. It is a remarkable achievement of Kuchař and Torre [3] to have achieved just that in the form of the Gaussian dust. We have applied essential steps of the analysis performed in [17] to the model [3] in order to show that all that was said in [17, 18, 21, 33] continues to be correct. We believe that in the class of models of type II considered, the degree of simplicity of the physical Hamiltonian cannot be improved over the model presented in section 4.

We believe that switching from the earlier operator constraint reduction (Dirac) approach that dominated the research in LQG over the past 20 years to the reduced phase space approach, concretely implemented within LQG for the first time in [17, 21] has many merits. Within the former approach, there are many steps of significant technical complexity to be overcome before one reaches the same level of technical and conceptual control as in the latter. After all, ultimately also in the former approach one aims at *the physical inner product, physical vector states, physical observables and the physical dynamics* which come for free in the latter approach. Moreover, on physical grounds the intrinsic description of the physics in terms of material reference systems makes a lot of sense, hence the approach advocated here and by many others before is well motivated.

There are certainly some possible caveats: For instance, the classical reduction assumes that certain objects such as  $\det(\partial S/\partial x)$  is everywhere non vanishing. As one can show, this is a gauge invariant condition but it imposes a non holonomic constraint on the phase space. Hence, the description is not valid in the full unconstrained phase space. On the other hand, these issues are maybe less important than one thinks as the material reference system serves a role quite similar to three of the four real degrees of freedom of the Higgs field which give mass to  $W^\pm$  and  $Z$  bosons: They are simply absorbed into those vector bosons thus making them local Isospin  $SU(2)$  singlets. Similarly here, the dust field is absorbed into the geometry and matter degrees of freedom, thus making them local  $\text{Diff}(M)$  singlets. Hence the material reference system is the gravitational counterpart of the Higgs effect.

The real physical question is whether scalar fields such as the Higgs field or the dust considered here exist in nature. The answer might quite well be negative. Although the dust field considered here and elsewhere in some sense make it an ideal dark matter candidate since it is only interacting with gravity and with itself, it may nevertheless be phenomenologically excluded. What we need is maybe a realistic dark matter candidate, to be found in some extension of the standard model, that can serve the purpose of deparametrisation. It would be perhaps most economic to isolate four degrees of freedom from the spacetime metric tensor for the purposes of deparametrisation but this is more difficult than with matter because one needs to construct scalars using covariant derivatives which makes such a construction non local and thus impractical.

In any case, dust fields and their generalisations as described in the present work provide a proof of principle that a reduced phase space quantisation approach to LQG not only works but moreover has many advantages over the Dirac operator constraint approach because it lifts all the structures previously found in LQG directly to the physical versus kinematical level. In view of these advantages, the other caveats mentioned are in our mind problems of lower priority.

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## A Constraint Analysis for the Gaussian Dust Model

In this section the constraint analysis of the Gaussian dust model is presented. From the Legendre transform we deduce the following primary constraints

$$z = \pi, \quad z^j = \pi^j, \quad Z = \Pi, \quad Z_a = \Pi_a, \quad R^j := \epsilon^{jkl} P_k W_l \quad (\text{A.1})$$

with the momenta  $\Pi = \Pi_a = 0$  for lapse and shift functions respectively. We notice that only two of the 3 constraints  $R^j$  are linearly independent because obviously  $W_j R^j = 0$ . In what follows we will assume that  $\sum_j W_j^2 > 0$  and  $\sum_j P_j^2 > 0$ . Then in a patch with  $W_3, P_3 \neq 0$  we can choose

$$\zeta_1 := P_1 - \frac{W_1}{W_3} P_3 = -R^2/W_3, \quad \zeta_2 := P_2 - \frac{W_2}{W_3} P_3 = R^1/W_3 \quad (\text{A.2})$$

as the independent ones which tell that all  $P_j$  are proportional to  $P_3$ . The velocity combinations that can be solved for in (4.3) are

$$[L_n T] = \frac{P_3}{W_3 \sqrt{\det(q)}}, \quad W_j [L_n S^j] = \frac{P}{\sqrt{\det(q)}} - \frac{\rho P_3}{W_3 \sqrt{\det(q)}} \quad (\text{A.3})$$

We compute the canonical Hamiltonian density using the abbreviation  $Q = \sqrt{\det(q)}$

$$\begin{aligned}
\mathcal{H}_{\text{GD,can}} &= [P\dot{T} + P_j\dot{S}^j + \pi\dot{\rho} + \pi^j\dot{W}_j + \Pi\dot{N} + \Pi_a\dot{N}^a - \mathcal{L}_{\text{GD}}]_{(A.3)} \\
&= \{N^a[PT_{,a} + P_jS_{,a}^j] + uz + u_jz^j + VZ + V^aZ_a + v^I\zeta_I + N(P[L_nT] + \frac{P_3}{W_3}[W_jL_nS^j]) \\
&\quad + NQ\{\frac{1}{2}\rho[-[LnT]^2 + q^{ab}T_{,a}T_{,b} + 1] + [-[L_nT][W_j[L_nS^j]] + q^{ab}T_{,a}[W_jS_{,b}^j]\}_{(A.3)} \\
&= N^a[PT_{,a} + P_jS_{,a}^j] + uz + u_jz^j + VZ + V^aZ_a + v^I\zeta_I + N(P\frac{P_3}{W_3Q} + \frac{P_3}{W_3}[\frac{P}{Q} - \frac{P_3\rho}{W_3Q}]) \\
&\quad + NQ\{\frac{1}{2}\rho[-[\frac{P_3}{W_3Q}]^2 + q^{ab}T_{,a}T_{,b} + 1] + [-[\frac{P_3}{W_3Q}][\frac{P}{Q} - \frac{P_3\rho}{W_3Q}] + q^{ab}T_{,a}[W_jS_{,b}^j]\} \\
&= N^a[PT_{,a} + P_jS_{,a}^j] + uz + u_jz^j + VZ + V^aZ_a + v^I\zeta_I + N\{\frac{PP_3}{W_3Q} - \frac{\rho}{2Q}[\frac{P_3}{W_3}]^2\} \\
&\quad + NQ\{\frac{1}{2}\rho[q^{ab}T_{,a}T_{,b} + 1] + q^{ab}T_{,a}[W_jS_{,b}^j]\} \\
&= N^a[PT_{,a} + P_jS_{,a}^j] + uz + u_jz^j + VZ + V^aZ_a + [v^I - N\frac{W_3}{P_3}Qq^{ab}T_{,a}S_{,b}^I]\zeta_I \\
&\quad + N\{\frac{PP_3}{W_3Q} - \frac{\rho}{2Q}[\frac{P_3}{W_3}]^2\} \\
&\quad + NQ\{\frac{1}{2}\rho[q^{ab}T_{,a}T_{,b} + 1] + \frac{W_3}{P_3}q^{ab}T_{,a}[P_jS_{,b}^j]\} \tag{A.4}
\end{aligned}$$

where we used the abbreviations

$$u = \dot{\rho}, \quad u_j = \dot{W}_j, \quad V = \dot{N}, \quad V^a = \dot{N}^a, \quad v^I = L_nS^I; \quad I = 1, 2 \tag{A.5}$$

The contributions to the canonical Hamiltonian from geometry and standard matter are  $N^aC_a + NC$  which has to be added to (A.4) in order to obtain the total canonical Hamiltonian

$$H_{\text{can}} = \int_{\sigma} d^3x (\mathcal{H}_{\text{GD,can}} + NC + N^aC_a) \tag{A.6}$$

Stability of the primary constraints yields on the constraint surface of the primary constraints

$$\begin{aligned}
C^{\text{tot}} &:= \{Z, H_{\text{can}}\} = C + \frac{1}{Q}[\frac{PP_3}{W_3} - \frac{\rho}{2}[\frac{P_3}{W_3}]^2] + Q[\frac{1}{2}\rho[q^{ab}T_{,a}T_{,b} + 1] + \frac{W_3}{P_3}q^{ab}T_{,a}[P_jS_{,b}^j] \\
C_a^{\text{tot}} &:= \{Z_a, H_{\text{can}}\} = C_a + PT_{,a} + P_jS_{,a}^j \\
\frac{N}{2}s &:= \{z, H_{\text{can}}\} = \frac{N}{2}\{-\frac{1}{Q}[\frac{P_3}{W_3}]^2 + Q[q^{ab}T_{,a}T_{,b} + 1]\} \\
&\quad \{z_I, H_{\text{can}}\} = -[v^I - N\frac{W_3}{P_3}Qq^{ab}T_{,a}S_{,b}^I]\frac{P_3}{W_3} \\
NK &:= \{z_3, H_{\text{can}}\} = [v^I - N\frac{W_3}{P_3}Qq^{ab}T_{,a}S_{,b}^I]\frac{W_IP_3}{W_3^2} + N\{-\frac{PP_3}{W_3^2Q} + \frac{\rho}{Q}\frac{P_3^2}{W_3^3} + Q\frac{1}{P_3}q^{ab}T_{,a}[P_jS_{,b}^j]\} \\
&\quad \{\zeta_I, H_{\text{can}}\} = u_I\frac{P_3}{W_3} + M_I \tag{A.7}
\end{aligned}$$

where we used that  $\{\zeta_I, \zeta_J\} = 0$  and  $M_I$  is some complicated and non vanishing expression whose explicit form will be of no further interest. The set of equations (A.7) has to vanish which is accomplished by choosing

$$v^I = N\frac{W_3}{P_3}Qq^{ab}T_{,a}S_{,b}^I, \quad u_I = -\frac{P_3}{W_3}M_I \tag{A.8}$$



and imposing the secondary constraints  $C^{\text{tot}}$ ,  $C_a^{\text{tot}}$ ,  $s$ ,  $K$ . Notice that due to (A.8)  $K$  simplifies to

$$K = -\frac{PP_3}{W_3^2 Q} + \frac{\rho}{Q} \frac{P_3^2}{W_3^3} + Q \frac{1}{P_3} q^{ab} T_{,a} [P_j S_{,b}^j] \quad (\text{A.9})$$

and the terms proportional to  $\zeta_I$  drop from the Hamiltonian.

Turning to the stabilisation of the secondary constraints, we add to  $C_a^{\text{tot}}$  the linear combination of the already stabilised primary constraints  $\pi_{\rho,a} + \pi^j W_{j,a} + \Pi N_{,a} + \Pi_b N_{,a}^b + (\Pi_a N^b)_{,b}$ . Then  $C_a^{\text{tot}}$  generates spatial diffeomorphisms on all variables involved and since  $\mathcal{H}_{\text{can}}$  is a spatial scalar density of weight one,  $C_a^{\text{tot}}$  is stabilised. Next we have on the constraint surface determined so far

$$\begin{aligned} 0 &= \{C^{\text{tot}}, H_{\text{can}}\} = \{C^{\text{tot}}, C^{\text{tot}}[N]\} - u \frac{\partial C^{\text{tot}}}{\partial \rho} - u^3 \frac{\partial C^{\text{tot}}}{\partial W_3} \\ &= \{C^{\text{tot}}, C^{\text{tot}}[N]\} - \frac{u}{2} s - u^3 K \\ &= \{C^{\text{tot}}, C^{\text{tot}}[N]\} \\ 0 &= \{s, H_{\text{can}}\} = -2u^3 \frac{1}{Q} \frac{P_3^2}{W_3^3} + M \\ 0 &= \{K, H_{\text{can}}\} = -u \frac{1}{Q} \frac{P_3^2}{W_3^3} + M' \end{aligned} \quad (\text{A.10})$$

where  $M$  is independent of  $u$  and  $M'$  depends linearly on  $u^3$ . We can therefore solve the two last equations for  $u, u^3$  respectively so that  $s, K$  are stabilised. As far as the first term is concerned we write for some smearing function  $f$

$$\begin{aligned} C_{\text{GD}}[f] &:= \int d^3x f (T + U) \\ &:= \int d^3x f \left\{ \frac{1}{Q} \left( \frac{PP_3}{W_3} - \frac{\rho}{2} \left[ \frac{P_3}{W_3} \right]^2 \right) + Q \left( \frac{1}{2} \rho [q^{ab} T_{,a} T_{,b} + 1] + \frac{W_3}{P_3} q^{ab} T_{,a} [P_j S_{,b}^j] \right) \right\} \end{aligned} \quad (\text{A.11})$$

and similar for  $C[f]$ . Then

$$\{C^{\text{tot}}[f], C^{\text{tot}}[f']\} = \{C[f], C[f']\} + \{C[f], C_{\text{GD}}[f']\} - \{C[f'], C_{\text{GD}}[f]\} + \{C_{\text{GD}}[f], C_{\text{GD}}[f']\} \quad (\text{A.12})$$

The first term gives  $-C_a [q^{ab} (f f'_{,b} - f_{,b} f')]$  as is well known from the hypersurface deformation algebra [30]. The second and first term cancel each other because the only piece from  $C$  that contributes is the gravitational piece which acts ultralocally only on the  $q_{ab}$  dependence in  $C_{\text{GD}}$  which thus gives due to the non derivative coupling a term proportional to  $\delta(x, y) [f(x) f'(y) - f(y) f'(x)] = 0$ . The last term is of a new type. Again due to vanishing ultralocal terms we just need to focus on terms that lead to derivatives of the  $\delta$  distributions. For the same reason we only need to keep track of the  $x, y$  dependence of the smearing fields. Accordingly in the

following calculation we neglect ultralocal contributions

$$\begin{aligned}
\{C_{\text{GD}}[f], C_{\text{GD}}[f']\} &= \frac{1}{2} \int d^3x \int d^3y [f(x)f'(y) - f(y)f'(x)] \{C_{\text{GD}}(x), C_{\text{GD}}(y)\} \\
&= \frac{1}{2} \int d^3x \int d^3y [f(x)f'(y) - f(y)f'(x)] \\
&\quad \times [\{T(x), T(y)\} + \{T(x), V(y)\} - \{T(y), V(x)\} + \{V(x), V(y)\}] \\
\{T(x), T(y)\} &= 0 \\
\{T(x), V(y)\} &= \frac{1}{W_3} \{ (PP_3)(x), (\frac{1}{2}\rho[q^{ab}T_{,a}T_{,b} + 1] + \frac{W_3}{P_3}q^{ab}T_{,a}[P_jS_{,b}^j])(y) \} \\
&\quad - \frac{\rho P_3}{W_3^2} \{ P_3(x), (\frac{1}{2}\rho[q^{ab}T_{,a}T_{,b} + 1] + \frac{W_3}{P_3}q^{ab}T_{,a}[P_jS_{,b}^j])(y) \} \\
&= \frac{P_3}{W_3} q^{ab}[\rho T_{,a} + \frac{W_3}{P_3}P_jS_{,a}^j] + \frac{P}{W_3} q^{ab}T_{,a} \frac{W_3}{P_3} P_j \delta_3^j \delta_{,y^b} \\
&\quad - \frac{\rho P_3}{W_3^2} \frac{W_3}{P_3} q^{ab}T_{,a} P_j \delta_3^j \delta_{,y^b} \\
&= \delta_{,y^b} q^{ab} [P_j S_{,a}^j + P T_{,a}] \\
\{V(x), V(y)\} &= Q^2 q^{ab} T_{,b} q^{cd} T_{,d} W_3^2 \{ (P_j S_{,b}^j / P_3)(x), (P_k S_{,d}^k / P_3)(y) \} \\
&= Q^2 q^{ab} T_{,b} q^{cd} T_{,d} W_3^2 S_{,b}^j \{ \frac{P_j}{P_3}(x), S_{,d}^k(y) \} \frac{P_k}{P_3} - x \leftrightarrow y \\
&= Q^2 q^{ab} T_{,b} q^{cd} T_{,d} W_3^2 S_{,b}^I \{ \frac{P_I}{P_3}(x), S_{,d}^k(y) \} \frac{P_k}{P_3} - x \leftrightarrow y \\
&= Q^2 q^{ab} T_{,b} q^{cd} T_{,d} W_3^2 S_{,b}^I [\{ P_I(x), S_{,d}^k(y) \} \frac{P_k}{P_3^2} - \{ P_3(x), S_{,d}^k(y) \} \frac{P_k P_I}{P_3^3}] - x \leftrightarrow y \\
&= Q^2 q^{ab} T_{,b} q^{cd} T_{,d} W_3^2 S_{,b}^I \delta_{,y^d} [\frac{P_I}{P_3^2} - \frac{P_3 P_I}{P_3^3}] - x \leftrightarrow y \\
&= 0
\end{aligned} \tag{A.13}$$

It follows

$$\begin{aligned}
\{C_{\text{GD}}[f], C_{\text{GD}}[f']\} &= \int d^3x \int d^3y [f(x)f'(y) - f(y)f'(x)] \{T(x), V(y)\} \\
&= \int d^3x \int d^3y [f(x)f'(y) - f(y)f'(x)] \delta_{,y^b} q^{ab} C_{\text{GD}a} \\
&= -C_{\text{GD}}[q^{ab}(f f'_{,b} - f_{,b} f')]
\end{aligned} \tag{A.14}$$

and therefore

$$\{C^{\text{tot}}[f], C^{\text{tot}}[f']\} = -C_a^{\text{tot}}[q^{ab}(f f'_{,b} - f_{,b} f')] \tag{A.15}$$

satisfies the hypersurface deformation algebra. All constraints are now stabilised.

The constraints  $Z, Z_a$  are trivially first class. The constraint  $C_a^{\text{tot}}$  is first class as we have already seen due to its geometrical action. The constraints  $z_I, \zeta_I$ ,  $I = 1, 2$  form second class partners as well as  $(z, K)$  and  $(z_3, s)$ . Finally, the constraint  $C^{\text{tot}}$  closes to itself with  $Z, Z_a, C_a^{\text{tot}}$  and with  $z, z_3$  respectively it closes to  $s, K$  respectively. To make it close with  $z_I, \zeta_I, s, K$  as well we add suitable linear combinations of all second class constraints to  $C^{\text{tot}}$  so that it has vanishing Poisson brackets with all second class constraints.

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